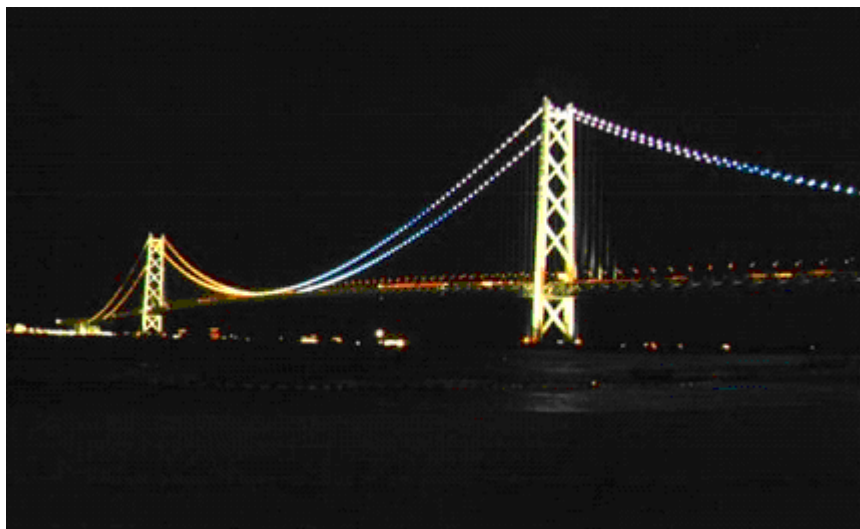


## Suspension Bridges

Another type of bridge is the suspension bridge. You may be familiar with the famous Golden Gate Bridge in San Francisco, CA, shown below.



Notice the shape of the main cable that goes from the top of the tower down to the span and back up to the next tower. Can you identify the shape of this cable?



The bridge above has the same feature. The shape is a parabola, just like the one studied in calculus. Another name for this shape is a **catenary**.

Recognizing this shape enables us to perform calculations to figure out measurements of the suspension bridge. Let us examine the mathematics of a catenary.

**Catenary:** The curve described by a uniform, flexible chain hanging under the influence of gravity.

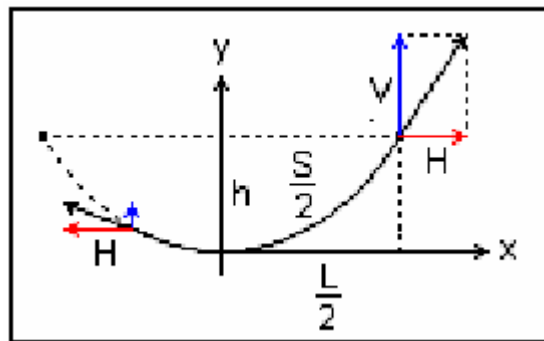


Figure 1. Catenary curve.

$S$  = Total Length of the Chain  
 $L$  = Total Length of the Span  
 $h$  = Sag  
 $w$  = Weight per unit Length

The tension  $T$  in the chain is in the direction of the derivative  $dy/dx$ . Its horizontal component is  $H$ , and its vertical component is  $V$ .  $H$  is constant at all points of the chain, since the weight of the chain acts vertically downward. Therefore, the vertical component of the tension in the chain is  $Hy'$ , where  $y'$  stands for  $dy/dx$ .

$T$  = tension in chain, in direction of the derivative  $dy/dx$   
 $H$  = Horizontal component of tension (constant)  
 $V = Hy' =$  Vertical component of tension ( $y' = dy/dx$ )

Consider the equilibrium of the short length of chain,  $dx$ .

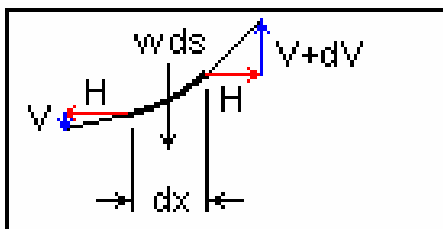


Figure 2. Equilibrium of length  $dx$  of chain.

The horizontal forces are equal, as we have already seen, and the sum of the vertical forces is

$$V + dV - V - wds = 0 \quad (1)$$

From

$$V = Hy' \quad (2)$$

We obtain

$$dV = Hy'' dx \quad (3)$$

Using the conditions that  $y = y' = 0$  at  $x = 0$ , we integrate to find the equation of the **Catenary** (4).

$$wds = Hy''$$

$$ds = \sqrt{1 + y'^2} dx$$

$$\int \frac{dy'}{\sqrt{1 + y'^2}} = \int_0^x \frac{w}{H} dx$$

$$\sinh^{-1} y' = \frac{w}{H} x$$

$$y' = \sinh \frac{w}{H} x$$

$$y = \frac{H}{w} \left( \cosh \frac{wx}{H} - 1 \right) \quad (4)$$

The catenary is a hyperbolic cosine curve, and its slope varies as the hyperbolic sine. The **arc length**, measured from the origin, is

$$s = \frac{H}{w} \sinh \frac{wx}{H} \quad (5)$$

which shows that the slope is proportional to arc length. The **total length of the chain** is

$$S = \frac{2H}{w} \sinh \frac{wL}{2H} \quad (6)$$

and

$$h = \frac{H}{w} (\cosh \frac{wL}{2H} - 1) \quad (7)$$

We can use these two equations to find that

$$H = \frac{w}{8h} (S^2 - 4h^2) \quad (8)$$

in terms of the total length of the chain and the sag. Then, the **span**,

$$L = \frac{2H}{w} \sinh^{-1} \frac{Sw}{2H} \quad (9)$$

The **vertical force** at a support is

$$V = \frac{wS}{2} \quad (10)$$

so the **tension** there is given by

$$T^2 = H^2 + \left(\frac{wS}{2}\right)^2 \quad (11)$$

Thus far, we have assumed that the chain is supported at equal heights on the two sides, but this is not necessary, and the equations can be modified to suit any case. If we expand the hyperbolic cosine in powers of  $x$ , the lowest term yields

$$y = \frac{wx^2}{2H}$$

Therefore, to this approximation, we have

$$h = \frac{wL^2}{8H}$$

a familiar result for the parabolic curve of a weightless chain supporting a uniform load, easily obtained by taking moments about the right-hand support of the chain. If we assume the weight is uniformly distributed in  $x$ , the differential equation is

$$wdx = Hy''$$

which, when integrated twice, gives just the result above (equation 4).

If a vertical downward force  $F$  acts at a point of the chain, the slope is discontinuous at that point, jumping by an amount  $F/H$ . This will add a term

$$F\delta(x - x_0)$$

for a force  $F$  at  $x = x_0$  to the differential equation. The shape of the whole chain changes, illustrating a well-known problem with suspension bridges. If the deck is not sufficiently rigid, the bridge can oscillate, one portion sinking as another rises. This can happen in opposite phase on the two sides of the bridge, resulting in a torsional oscillation.

If the chain is imagined to become rigid, and is turned over, tensions will become compressions, and we will have an arch. The curve shows how the stress is propagated to the supports, provided the structure remains rigid. In this case, a thin structure tends to buckle, a problem not encountered with the chain. The results must be used with care for arches, since the loading conditions are usually quite different. The usual circular arch has heavy haunches, which helps to guide the force along the voussoirs. A segmental arch is helped by pierced spandrels, since this reduces the force on a flatter portion of the arch and distributes the load more uniformly.

Let us try some exercises using the catenary.

The main span of a certain suspension bridge is horizontal and 1000 feet long. It is suspended from two main cables hanging between two 400 foot towers, so that these cables just touch the roadway in the center of the span. The main cables each form a parabolic arc going from the top of one tower down to the roadway and up to the top of the other tower.

- a) Write but do not solve the formula for the total length of one of the main cables.
- b) Calculate the horizontal component of the tension in the chain.
- c) Calculate the vertical force at a support.
- d) Calculate the tension at a support.